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Two results about pseudo-differential
operators.

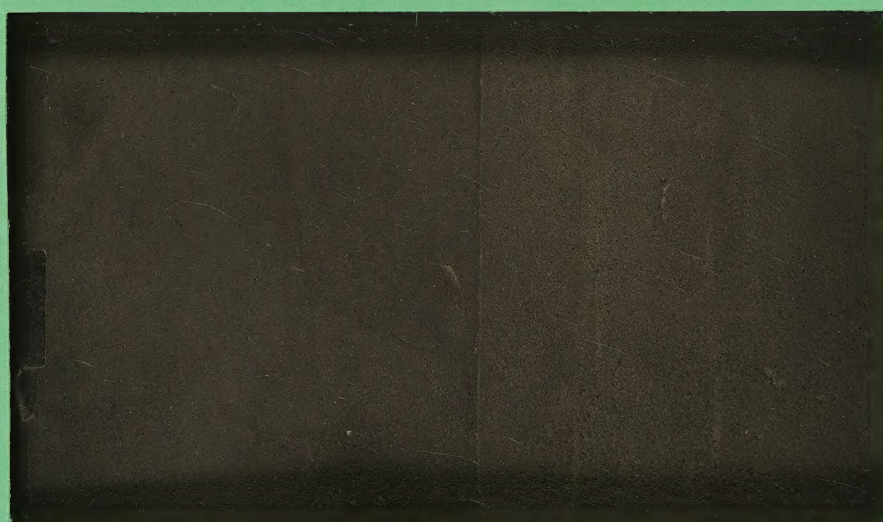
by Anders Melin

fil.kand. Smålands nation

1973

DEPARTMENT OF MATHEMATICS

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This dissertation consists of a summary of the following two papers.

- [A] Melin, A., Lower bounds for pseudo-differential operators. Ark. för Matematik, 9 (1971), 117 - 140.
- [B] Melin, A., A remark on invariant pseudo-differential operators. Math. Scand, 30 (1972), 290 - 296.

Lower bounds for pseudo-differential operators.

Let $P = P(x, D)$, $D = \partial/i\partial x$, be a classical pseudo-differential operator in an open set Ω in R^n of order m and with a non-negative principal symbol p_m . Then as was proved by Gårding [3] (for differential operators) one has estimates of P from below of the form

$$(1) \quad \operatorname{Re} (Pu, u) + \epsilon |u|_{(\mu)}^2 \geq C_{K, \epsilon, s} |u|_{(s)}^2, \quad u \in C_0^\infty(K),$$

with $\mu = m/2$. That is, for every $\epsilon > 0$ and every $s \in R$, the inequality (1) is valid for all C^∞ functions with support in the compact set K in Ω when the constant $C_{K, \epsilon, s}$ is suitably chosen. Here $|u|_{(s)}$ denotes the norm of u in the space of functions with derivatives of order s in L^2 .

Gårding's result is nowadays easily obtained from the calculus of pseudo-differential operators which gives the same result even for every $\mu > (m-1)/2$. The corresponding inequalities with $\mu = (m-1)/2$ and arbitrarily small ϵ are not

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valid in general without additional restriction on the term p_{m-1} of order $m-1$ of the symbol. It was proved by Hörmander [4] that (1) is valid with $\mu = (m-1)/2$ if ϵ is replaced by some positive constant depending on K (the sharp Gårding inequality).

In [A] we give a necessary and sufficient condition on P for (1) to hold for every $\epsilon > 0$ when $\mu = (m-1)/2$. We use the localization technique introduced by Hörmander in [4] and [5] and prove after reduction to the case $m=1$ that a necessary and sufficient condition is that

$$(2) \quad \operatorname{Re} \int \overline{v(x)} \left(\sum_{|\alpha+\beta|=2} p_{1\beta}^{\alpha}(y, \eta) x^{\beta} D^{\alpha} v(x) / \alpha! \beta! \right) dx +$$

$$+ \operatorname{Re} p_0(y, \eta) \int |v(x)|^2 dx \geq 0, \quad v \in C_0^{\infty}(\mathbb{R}^n),$$

$$p_{1\beta}^{\alpha} = (iD_{\xi})^{\alpha} (iD_x)^{\beta} p_1,$$

when $(y, \eta) \in \Omega \times S^{n-1}$ is a zero for p_1 .

This is carried out in Section 3. In Section 2 we give necessary and sufficient conditions on the complex coefficients a_{β}^{α} , $|\alpha+\beta| \leq 2$, for the inequality

$$(3) \quad \operatorname{Re} \int \overline{v(x)} \left(\sum_{|\alpha+\beta| \leq 2} a_{\beta}^{\alpha} x^{\beta} D^{\alpha} v(x) / \alpha! \beta! \right) dx \geq 0, \quad v \in C_0^{\infty}(\mathbb{R}^n).$$

By using the invariance of (3) when (x, D) is treated as a vector in $\mathbb{R}^n \oplus \mathbb{R}^n$ and is transformed by the symplectic group, we are able to put the inequality (3) into a diagonal form, thereby reducing to the classical uncertainty relation. This can be done when the form

$$h(y, \eta) = \sum_{|\alpha+\beta|=2} a_{\beta}^{\alpha} y^{\beta} \eta^{\alpha} / \alpha! \beta!, \quad (y, \eta) \in \mathbb{R}^n \oplus \mathbb{R}^n$$

is positive definite. To each such positive semi-definite form we associate a certain trace denoted $\widetilde{\text{Tr}}(h)$ and defined as the sum of the positive elements in i Spectrum (JH) where H is the corresponding matrix for h and $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, I denoting the identity matrix. Then combining Section 2 and 3 we can prove that a necessary and sufficient condition for (1) to hold for every $\varepsilon > 0$ and every $s \in \mathbb{R}$ is that

$$\text{Re } p_{m-1} + \widetilde{\text{Tr}}(H_{p_m}) \geq 0$$

at the zeros of p_m in $\Omega \times S^{n-1}$, H_{p_m} denoting the Hessian of p_m .

In Section 4 we prove a result due to Radkevič [7] stating that a certain class of pseudo-differential operators with non-negative principal symbols is hypoelliptic. Our results have been used by Fujiwara-Uchiyama [2] to establish a priori estimates for some boundary value problems for the Laplacian. We also remark that we have not treated the case where u in (1) is vector-valued. As was proved by Lax-Nirenberg [6] (see also [1] and [10]) one still has a sharp Gårding inequality in that case when the principal symbol is a positive semi-definite matrix.

A remark on invariant pseudo-differential operators.

In [B] we characterize those connected Lie groups G which have the property:

(*) every bi-invariant pseudo-local operator on G is the sum of a differential operator and an operator with a smooth kernel.

We prove that G has the property (*) if and only if the Lie algebra of G is not compact. The corresponding result with pseudo-local operators replaced by classical pseudo-differential operators was proved by Stetkaer [9]. As Stetkaer pointed out every bi-invariant pseudo-local operator reduces to a differential operator if every bi-invariant C^∞ function in $G \setminus e$ can be extended to a C^∞ function on G . By using the existence of non-trivial nilpotent elements in a non-compact Lie algebra we reduce the problem to proving that a function in $C^\infty(\mathbb{R}^n \setminus 0)$ which is invariant under the one-parameter group $\exp(tN)$ generated by a nilpotent matrix $N \neq 0$ can be extended to $C^\infty(\mathbb{R}^n)$.

Finally we remark that part of our result has recently been proved by Rothschild [8] who proved that (*) is valid if the Lie algebra of G is not the semi-direct sum of a compact and a solvable algebra.

I wish to express my great gratitude to my teachers of mathematics.

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